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Time-periodic solution for a fourth-order parabolic equation describing crystal surface growth*

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Abstract. In this paper, by using the Galerkin method, the existence and uniqueness of time-periodic generalized solutions to a fourth-order parabolic equation describing crystal surface growth are proved.
Keywords. Time-periodic solution, fourth-order parabolic equation, Galerkin method.

1 Introduction

In the study of crystal surface growth, there arises the following diffusion equation

$$u_t = -j_x + f(x,t),$$

where $u(x,t)$ denotes the variation of height from the average, j is the atom current parallel to the surface, and $f(x,t)$ is a noise term caused by shot noise in the incoming flux. Taking $j = u_{xxx} + \frac{u_x}{1+|u_x|^2}$, we obtain the well-known BCF model (see [4, 5, 7, 8, 11, 13])

$$u_t + u_{xxxx} + \left(\frac{u_x}{1+|u_x|^2}\right)_x = f(x,t), \quad \text{in } (0,1)\times\mathbf{R}. \tag{1.1}$$

During the past years, many authors have paid much attention to the equation (1.1). It was Rost and Krug [13] who studied the unstable epitaxy on singular surfaces using equation (1.1) with a prescribed slopedependent surface current. In their paper, they derived scaling relations for the late stage of growth, where power law coarsening of the mound morphology is observed. In [11], in the limit of weak desorption, O. Pierre-Louis et al. derived the equation (1.1) for a vicinal surface growing in the step flow mode. This limit turned out to be singular, and nonlinearities of arbitrary order need to be taken into account. Recently, H. Fujimura and A. Yagi [4] considered the equation of (1.1). In their paper, the uniqueness local solutions and the global solutions are obtained, a dynamical system determined from the initial-boundary value problem of the model equation was also constructed. In [5], H. Fujimura and A. Yagi continued a study on the model equation (1.1). They considered the asymptotic behavior of trajectories of the dynamical system by constructing exponential attractors and a Lyapunov function. There is much literature concerned with the Eq.(1.1), for more results we refer the reader to [6, 14] and the references therein.

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Furthermore, several authors have paid attention to the time-periodic problems [1, 19, 20]. But, to the best of our knowledge, only a few papers deal with time periodic solutions of fourth-order diffusion equations. In [10, 17], the existence of time periodic solutions for the Cahn-Hilliard type equation and viscous Cahn-Hilliard equation with periodic concentration dependent potentials and sources has been investigated. In [15, 16], Wang et. al. considered the existence and uniqueness of time-periodic generalized solutions and time-periodic classical solutions to the generalized Ginzburg-Landau model equation in 1D and 2D case. In [3], by using the Galerkin method and the Leray-Schauder fixed point theorem, Fu and Guo studied the existence and uniqueness of a time periodic solution for the viscous Camassa-Holm equation. There are also many papers were denoted to the periodic problems, for example [9, 12, 18] and so on.

Here, we investigate the existence and uniqueness of time-periodic generalized solutions to the equation (1.1) in one spatial dimension together with the condition

$$u(x+1,t) = u(x,t),\ t \in \mathbf{R}, \tag{1.2}$$

and the time-periodic condition

$$u(x,t+\omega) = u(x,t), \quad x \in (0,1),\ t \in \mathbf{R}, \tag{1.3}$$

where $\omega > 0$ is a constant and $f(x,t)$ is ω-periodic functions with respect to the time t, which also satisfies

$$\int_\Omega f(x,t)dx = 0.$$

Throughout this paper, we use the following notations.

Let X be a Banach space, $C^k_\omega(\mathbf{R};X)$ denotes the set of X-valued ω-periodic functions on $\mathbf{R}$ with continuous derivatives up to order k. The norm in $C^k_\omega(\mathbf{R};X)$ is defined as

$$\|u\|_{C^k_\omega(\mathbf{R};X)} = \sup_{0\le t\le\omega}\{\sum_{i=0}^k \|D^i_t u\|_X\},$$

where $D_t = \frac{\partial}{\partial t}$, $\|\cdot\|_X$ is the norm in X. We also define $L^p_\omega(\mathbf{R};X)$ $(1 \le p \le \infty)$ as the set of ω-periodic X-valued measurable functions on $\mathbf{R}$ such that

$$\|u\|_{L^p_\omega(\mathbf{R};X)} = \left(\int_0^\omega \|u\|^p_X dt\right)^{\frac{1}{p}} < \infty, \quad \text{where } 1 \le p < \infty,$$

$$\|u\|_{L^p_\omega(\mathbf{R};X)} = \operatorname{ess}\sup_{0\le t\le\omega} \|u\|_X < \infty, \quad \text{where } p = \infty.$$

Let $W^{k,p}_\omega(\mathbf{R};X)$ denote the set of functions which belong to $L^p_\omega(\mathbf{R};X)$ together with their partial derivatives with respect to t up to the order k.

In the following, we frequently use the Poincaré inequality (see [2]):

$$\|u\|^2 \le \frac{1}{2}\|u_x\|^2, \quad \text{where } \int_0^1 u(x,t)dx = 0.$$

Denote $\|\cdot\|_{L^2(0,1)}$ by $\|\cdot\|$, $\|\cdot\|_{L^\infty(0,1)}$ by $\|\cdot\|_\infty$, $\|\cdot\|_{L^p(0,1)}$ by $\|\cdot\|_p$ and $\|\cdot\|_{H^m(0,1)}$ by $\|\cdot\|_{H^m}$, respectively.

2 Integration estimations and existence of the approximate solutions for problem (1.1)-(1.3)

Let $\{y_j(x)\}$ $(j = 1, 2, \cdots)$ be the orthonormal base in $L^2(0,1)$ being composed of the eigenfunctions of the eigenvalue problem

$$y'' + \lambda y = 0, \quad y'(0) = y'(1) = 0,$$

corresponding to eigenvalues λ_j $(j = 1, 2, \cdots)$.

Suppose that $u_N(x,t) = \sum_{j=1}^{N} u_{Nj}(t)y_j(x)$ is the Galerkin approximate solution to the problem (1.1)-(1.3), where a group of function $u_{Nj}(t)$ $(j = 1, 2, \cdots, N) \in C^1(\omega, \mathbf{R})$, N is a natural number.

Performing the Galerkin procedure for the equation (1.1), we obtain

$$u_{Nt} + u_{Nxxxx} + \left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)_x = f, \quad \text{in } (0,1)\times\mathbf{R}. \tag{2.1}$$

with

$$u_N(x+1,t) = u_N(x,t), \quad t \in \mathbf{R}, \tag{2.2}$$

and the time-periodic condition

$$u_N(x, t+\omega) = u_N(x,t), \quad x \in (0,1), \ t \in \mathbf{R}, \tag{2.3}$$

Lemma 2.1 *Suppose that $f \in C_\omega(\mathbf{R}; L^2(0,1))$, $M_1 = \sup_{0\le t\le\omega}\|f(\cdot,t)\|$. Then, there exists a approximate solution u_N for problem (2.1)-(2.3), which satisfies*

$$\sup_{0\le t\le\omega}\|u_N(\cdot,t)\|^2 \le c_0 M_1^2,$$

where c_0 is a positive constant independent of N, M_1.

Proof. Using Poincaré's inequality and Hölder's inequality, we have

$$\|u_N\|^2 \le \frac{1}{2}\|u_{Nx}\|^2 = -\frac{1}{2}\int_0^1 u_N u_{Nxx}dx \le \frac{1}{4}\|u_N\|^2 + \frac{1}{4}\|u_{Nxx}\|^2,$$

that is

$$\|u_N\|^2 \le \frac{1}{3}\|u_{Nxx}\|^2. \tag{2.4}$$

Multiplying both sides of (2.1) by u_N, and integrating it over $(0,1)$, making use of Hölder's inequality and (2.4), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_N\|^2 + \|u_{Nxx}\|^2 \\
= &\int_0^1 \frac{u_{Nx}^2}{1+|u_{Nx}|^2}dx + \int_0^1 f u_N dx \le \|u_{Nx}\|^2 + \frac{1}{8}\|u_N\|^2 + 2\|f\|^2 \\
\le &\frac{1}{2}\|u_{Nxx}\|^2 + \frac{5}{8}\|u_N\|^2 + 2\|f\|^2 \le \frac{17}{24}\|u_{Nxx}\|^2 + 2M_1^2.
\end{aligned}$$

Then,

$$\frac{d}{dt}\|u_N\|^2 + \frac{7}{12}\|u_{Nxx}\|^2 \le 4M_1^2. \tag{2.5}$$

Integrating (2.5) over $[0,\omega]$, we get

$$\int_0^\omega \|u_{Nxx}\|^2 \le \frac{48}{7}M_1^2\omega. \tag{2.6}$$

It then follows from (2.6) that there is a $t_1 \in (0,\omega)$ such that

$$\|u_{Nxx}(\cdot,t_1)\|^2 \le \frac{48}{7}M_1^2. \tag{2.7}$$

Adding (2.4) and (2.7) together gives

$$\|u_N(\cdot,t_1)\|^2 \le \frac{16}{7}M_1^2. \tag{2.8}$$

Integrating (2.5) again over $[t_1, t+\omega]$ ($\forall t \in [0,\omega]$), using (2.8), we get

$$\sup_{0\le t\le\omega} \|u_N(\cdot,t)\|^2 \le \|u_N(\cdot,t_1)\|^2 + 8\omega M_1^2 = (\frac{16}{7} + 8\omega)M_1^2. \tag{2.9}$$

Setting $c_0 = \frac{16}{7} + 8\omega$, we complete the proof.

Employing the Leray-Schauder fixed-point argument, we can prove that there exists at least one solution $u_N(t) = \sum_{j=1}^N u_{Nj}y_j(x)$ for the problem (2.1)-(2.3).

Lemma 2.2 *Suppose that the assumptions of Lemma 2.1 hold and*

$$f \in C_\omega(\mathbf{R}; H^2(0,1)), f_t \in C_\omega(\mathbf{R}; L^2(0,1)).$$

Then

$$\sup_{0\le t\le\omega} (\|u_{Nt}(\cdot,t)\|_{H^2}^2 + \|u_N(\cdot,t)\|_{H^4}^2) \le c_1(M_2),$$

where $M_2 = \sup_{0\le t\le\omega}(\|f\|_{H^2} + \|f_t\|)$. *Here and in the sequel,* $c_i(M_2)(i = 1,2,\cdots)$ *is nondecreasing with respect to* M_2 *and* $\lim_{M_2\to 0} c_i(M_2) = c_i(0) = 0$, $c_i(M_2)$ *is independent of* N.

Proof. Multiplying both sides of (2.1) by $-u_{Nxx}$, and integrating it over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{Nx}\|^2 + \|u_{Nxxx}\|^2 + \int_0^1 \frac{u_{Nx}u_{Nxxx}}{1+|u_{Nx}|^2}dx + \int_0^1 fu_{Nxx}dx = 0,$$

Then, using Hölder's inequality, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_{Nx}\|^2 + \|u_{Nxxx}\|^2 \\ \le\ & \frac{1}{4}\|u_{Nxxx}\|^2 + \int_0^1 \frac{u_{Nx}^2}{(1+|u_{Nx}|^2)^2}dx + \frac{1}{4}\|f\|^2 + \|u_{Nxx}\|^2 \\ \le\ & \frac{1}{4}\|u_{Nxxx}\|^2 + \|u_{Nx}\|^2 + \frac{1}{4}\|f\|^2 + \|u_{Nxx}\|^2. \end{aligned}$$

By Nirenberg's inequality, we have

$$\|u_{Nx}\|^2 \le \left(c_1'\|u_{Nxxx}\|^{\frac{1}{3}}\|u_N\|^{\frac{2}{3}} + c_2'\|u_N\|\right)^2 \le \frac{1}{8}\|u_{Nxxx}\|^2 + c_2(M_2).$$

and

$$\|u_{Nxx}\|^2 \le \left(c_1'\|u_{Nxxx}\|^{\frac{2}{3}}\|u_N\|^{\frac{1}{3}} + c_2'\|u_N\|\right)^2 \le \frac{1}{8}\|u_{Nxxx}\|^2 + c_3(M_2).$$

Summing up, we get

$$\frac{d}{dt}\|u_{Nx}\|^2 + \|u_{Nxxx}\|^2 \le c_4(M_2), \tag{2.10}$$

where $c_4(M_2) = 2c_2(M_2) + 2c_3(M_2) + \frac{1}{2}M_2^2$. Integrating (2.10) over $[0,\omega]$, we have

$$\int_0^\omega \|u_{Nxxx}(\cdot,t)\|^2 dt \le c_4(M_2)\omega. \tag{2.11}$$

It then follows from (2.11) that there exists a time $t_2 \in (0,\omega)$ such that

$$\|u_{Nxxx}(\cdot,t_2)\|^2 \le c_4(M_2). \tag{2.12}$$

Then

$$\|u_{Nx}(\cdot,t_2)\|^2 \le \frac{1}{8}\|u_{Nxxx}(\cdot,t_2)\|^2 + c_2(M_2) \le \frac{1}{8}c_4(M_2) + c_2(M_2). \tag{2.13}$$

Integrating (2.10) again over $[t_2, t+\omega]$ ($\forall t \in [0,\omega]$), using (2.13), we obtain

$$\sup_{0\le t\le\omega} \|u_{Nx}(\cdot,t)\|^2 \le \|u_{Nx}(\cdot,t_2)\|^2 + 2c_4(M_2)\omega = c_5(M_2). \tag{2.14}$$

Based on Sobolev's embedding theorem, we have

$$\|u_N(\cdot,t)\|_{C[0,1]} \le c\|u_N\|_{H^1} \le c_6(M_2), \quad t \in [0,\omega]. \tag{2.15}$$

Multiplying both sides of (2.1) by u_{Nxxxx}, and integrating it over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{Nxx}\|^2 + \|u_{Nxxxx}\|^2 = \int_0^1 \left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)_x u_{Nxxxx}dx + \int_0^1 f u_{Nxxxx}dx.$$

Then, using Hölder's inequality, noticing that $s \le \frac{1}{2}(1+s^2)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_{Nxx}\|^2 + \|u_{Nxxxx}\|^2 \\
\le\ & \frac{1}{4}\|u_{Nxxxx}\|^2 + \int_0^1 \left[\left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)\right]^2 dx + \frac{1}{4}\|u_{Nxxxx}\|^2 + \int_0^1 f^2 dx
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\|u_{Nxxxx}\|^2 + \int_0^1 \left[\frac{u_{Nxx}}{1+|u_{Nx}|^2} - \frac{2|u_{Nx}|^2 u_{Nxx}}{(1+|u_{Nx}|^2)^2}\right]^2 dx + \int_0^1 f^2 dx \\
&\le \frac{1}{2}\|u_{Nxxxx}\|^2 + 2\int_0^1 \left(\frac{u_{Nxx}}{1+|u_{Nx}|^2}\right)^2 dx \\
&\quad +2\int_0^1 \left(\frac{2|u_{Nx}|^2 u_{Nxx}}{(1+|u_{Nx}|^2)^2}\right)^2 dx + \int_0^1 f^2 dx \\
&= \frac{1}{2}\|u_{Nxxxx}\|^2 + 2\int_0^1 u_{Nxx}^2 \frac{1}{(1+|u_{Nx}|^2)^2} dx \\
&\quad +8\int_0^1 u_{Nxx}^2 \frac{|u_{Nx}|^4}{(1+|u_{Nx}|^2)^4} dx + \int_0^1 f^2 dx \\
&\le \frac{1}{2}\|u_{Nxxxx}\|^2 + 2\|u_{Nxx}\|^2 + \frac{1}{2}\|u_{Nxx}\|^2 + M^2.
\end{aligned}
$$

Using Nirenberg's inequality, we derive that

$$
\frac{5}{2}\|u_{Nxx}\|^2 \le \frac{5}{2}(c_1'\|u_{Nxxxx}\|^{\frac{1}{3}}\|u_{Nx}\|^{\frac{2}{3}} + c_2'\|u_{Nx}\|)^2 \le \frac{1}{4}\|u_{Nxxxx}\|^2 + c_7(M_2),
$$

Summing up, we deduce that

$$
\frac{d}{dt}\|u_{Nxx}\|^2 + \frac{1}{2}\|u_{Nxxxx}\|^2 \le c_8(M_2), \tag{2.16}
$$

where $c_8(M_2) = 2c_7(M_2) + 2M_2^2$. Integrating (2.16) over $[0, \omega]$, we have

$$
\int_0^\omega \|u_{Nxxxx}(\cdot, t)\|^2 dt \le 2c_8(M_2)\omega. \tag{2.17}
$$

It then follows from (2.17) that there exists a time $t_3 \in (0, \omega)$ such that

$$
\|u_{Nxxxx}(\cdot, t_3)\|^2 \le 2c_8(M_2). \tag{2.18}
$$

Then

$$
\|u_{Nxx}(\cdot, t_3)\|^2 \le \frac{1}{10}\|u_{Nxxxx}(\cdot, t_3)\|^2 + \frac{2}{5}c_7(M_2) \le \frac{1}{5}c_8(M_2) + \frac{2}{5}c_7(M_2). \tag{2.19}
$$

Integrating (2.16) again over $[t_3, t+\omega]$ ($\forall t \in [0, \omega]$), using (2.19), we obtain

$$
\sup_{0\le t\le\omega} \|u_{Nxx}(\cdot, t)\|^2 \le \|u_{Nxx}(\cdot, t_3)\|^2 + 2c_8(M_2)\omega = c_9(M_2). \tag{2.20}
$$

Based on Sobolev's embedding theorem, we have

$$
\|u_{Nx}(\cdot, t)\|_{C[0,1]} \le c\|u_N\|_{H^2} \le c_{10}(M_2), \quad t \in [0, \omega]. \tag{2.21}
$$

Multiplying both sides of (2.1) by $u_{Nxxxxxx}$, and integrating it over $(0, 1)$, we obtain

$$
\frac{1}{2}\frac{d}{dt}\|u_{Nxxx}\|^2 + \|u_{Nxxxxx}\|^2 + \int_0^1 \left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)_{xx} u_{Nxxxxxx} dx = \int_0^1 f_x u_{Nxxxxx} dx.
$$

Using Hölder's inequality, we get

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|u_{Nxxx}\|^2 + \|u_{Nxxxxx}\|^2 \\
\leq\ & \int_0^1 \left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)^2_{xx} dx + \int_0^1 f_x^2 dx + \frac{1}{2}\|u_{Nxxxxx}\|^2 \\
\leq\ & 4\int_0^1 \frac{u^2_{Nxxx}}{(1+|u_{Nx}|^2)^2}dx + 144\int_0^1 \frac{u^2_{Nx}u^4_{Nxx}}{(1+|u_{Nx}|^2)^4}dx + 16\int_0^1 \frac{u^4_{Nx}u^2_{Nxxx}}{(1+|u_{Nx}|^2)^4}dx \\
& +256\int_0^1 \frac{u^6_{Nx}u^4_{Nxx}}{(1+|u_{Nx}|^2)^6}dx + \int_0^1 f_x^2 dx + \frac{1}{2}\|u_{Nxxxxx}\|^2 \\
\leq\ & 5\|u_{Nxxx}\|^2 + 40\|u_{Nxx}\|_4^4 + M_2^2 + \frac{1}{2}\|u_{Nxxxxx}\|^2.
\end{aligned}
$$

By Nirenberg's inequality, we have

$$
5\|u_{Nxxx}\|^2 \leq 5(c_1'\|u_{Nxxxxx}\|^{\frac{1}{3}}\|u_{Nxx}\|^{\frac{2}{3}} + c_2'\|u_{Nxx}\|)^2 \leq \frac{1}{8}\|u_{Nxxxxx}\|^2 + c_{11}(M_2),
$$

and

$$
40\|u_{Nxx}\|_4^4 \leq 40(c_1'\|u_{Nxxxxx}\|^{\frac{1}{12}}\|u_{Nxx}\|^{\frac{11}{12}} + c_2'\|u_{Nxx}\|)^4 \leq \frac{1}{8}\|u_{Nxxxxx}\|^2 + c_{12}(M_2).
$$

Summing up, we deduce that

$$
\frac{d}{dt}\|u_{Nxxx}\|^2 + \frac{1}{2}\|u_{Nxxxxx}\|^2 \leq c_{13}(M_2), \tag{2.22}
$$

where $c_{13}(M_2) = 2c_{11}(M_2) + 2c_{12}(M_2) + 2M_2^2$. Integrating (2.22) over $[0,\omega]$, we have

$$
\int_0^\omega \|u_{Nxxxxx}(\cdot,t)\|^2 dt \leq 2c_{13}(M_2)\omega. \tag{2.23}
$$

It then follows from (2.23) that there exists a time $t_4 \in (0,\omega)$ such that

$$
\|u_{Nxxxxx}(\cdot,t_4)\|^2 \leq 2c_{13}(M_2). \tag{2.24}
$$

Then

$$
\begin{aligned}
\|u_{Nxxx}(\cdot,t_4)\|^2 &\leq \frac{1}{40}\|u_{Nxxxxx}(\cdot,t_4)\|^2 + \frac{1}{5}c_{11}(M_2) \\
&\leq \frac{1}{20}c_{13}(M_2) + \frac{1}{5}c_{11}(M_2).
\end{aligned} \tag{2.25}
$$

Integrating (2.22) again over $[t_4, t+\omega]$ ($\forall t \in [0,\omega]$), using (2.25), we obtain

$$
\sup_{0\leq t\leq\omega} \|u_{Nxxx}(\cdot,t)\|^2 \leq \|u_{Nxxx}(\cdot,t_4)\|^2 + 2c_{13}(M_2)\omega = c_{14}(M_2). \tag{2.26}
$$

Based on Sobolev's embedding theorem, we have

$$
\|u_{Nxx}(\cdot,t)\|_{C[0,1]} \leq c\|u_N\|_{H^3} \leq c_{15}(M_2), \quad t \in [0,\omega]. \tag{2.27}
$$

Multiplying both sides of (2.1) by $u_{Nxxxxxxxx}$, and integrating it over $(0,1)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_{Nxxxx}\|^2+\|u_{Nxxxxxx}\|^2+\int_0^1\left(\frac{u_{Nx}}{1+|u_{Nx}}^2\right)_{xxx}u_{Nxxxxxx}dx\\
=\ &\int_0^1 f_{xx}u_{Nxxxxxx}dx.
\end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_{Nxxxx}\|^2+\|u_{Nxxxxxx}\|^2\\
\leq\ &\frac{1}{2}\|u_{Nxxxxxx}\|^2+\int_0^1\left(\frac{u_{Nx}}{1+|u_{Nx}}^2\right)^2_{xxx}dx+\int_0^1 f_{xx}^2dx\\
\leq\ &\frac{1}{2}\|u_{Nxxxxxx}\|^2+\int_0^1 f_{xx}^2dx+c\int_0^1\left(\frac{u_{Nxxxx}}{1+|u_{Nx}|^2}\right)^2dx\\
&+c\int_0^1\left(\frac{u_{Nx}u_{Nxx}u_{Nxxx}}{(1+|u_{Nx}|^2)^2}\right)^2dx+c\int_0^1\left(\frac{|u_{Nxx}|^3}{(1+|u_{Nx}|^2)^2}\right)dx\\
&+c\int_0^1\left(\frac{|u_{Nx}|^2u_{Nxx}}{(1+|u_{Nx}|^2)^2}\right)dx+c\int_0^1\left(\frac{|u_{Nx}|^2|u_{Nxx}|^3}{(1+|u_{Nx}|^2)^3}\right)dx\\
&+c\int_0^1\left(\frac{|u_{Nx}|^3u_{Nxx}u_{Nxxx}}{(1+|u_{Nx}|^2)^3}\right)dx+c\int_0^1\left(\frac{|u_{Nx}|^4|u_{Nxx}|^3}{(1+|u_{Nx}|^2)^4}\right)dx\\
\leq\ &\frac{1}{2}\|u_{Nxxxxxx}\|^2+\int_0^1 f_{xx}^2dx+c(\|u_{Nxxxx}\|^2+\|u_{Nxx}\|_4^4\\
&+\|u_{Nxx}\|_6^6+\|u_{Nxxx}\|_4^4+\|u_{Nxx}\|^2).
\end{aligned}$$

By Nirenberg's inequality, we get

$$\begin{aligned}
c\|u_{Nxxxx}\|^2\ &\leq\ c(c_1'\|u_{Nxxxxxx}\|^{\frac{1}{3}}\|u_{Nxxx}\|^{\frac{2}{3}}+c_2'\|u_{Nxxx}\|)^2\\
&\leq\ \frac{1}{16}\|u_{Nxxxxxx}\|^2+c_{16}(M_2),
\end{aligned}$$

$$\begin{aligned}
c\|u_{Nxx}\|_4^4\ &\leq\ c(c_1'\|u_{Nxxxxxx}\|^{\frac{1}{16}}\|u_{Nxx}\|^{\frac{15}{16}}+c_2'\|u_{Nxx}\|)^4\\
&\leq\ \frac{1}{16}\|u_{Nxxxxxx}\|^2+c_{17}(M_2),
\end{aligned}$$

$$\begin{aligned}
c\|u_{Nxx}\|_6^6\ &\leq\ c(c_1'\|u_{Nxxxxxx}\|^{\frac{1}{12}}\|u_{Nxx}\|^{\frac{11}{12}}+c_2'\|u_{Nxx}\|)^6\\
&\leq\ \frac{1}{16}\|u_{Nxxxxxx}\|^2+c_{18}(M_2),
\end{aligned}$$

and

$$\begin{aligned}
c\|u_{Nxxx}\|_4^4\ &\leq\ c(c_1'\|u_{Nxxxxxx}\|^{\frac{1}{12}}\|u_{Nxxx}\|^{\frac{11}{12}}+c_2'\|u_{Nxxx}\|)^2\\
&\leq\ \frac{1}{16}\|u_{Nxxxxxx}\|^2+c_{19}(M_2).
\end{aligned}$$

Summing up, we deduce that

$$\frac{d}{dt}\|u_{Nxxxx}\|^2 + \frac{1}{2}\|u_{Nxxxxxx}\|^2 \le c_{21}(M_2), \tag{2.28}$$

where $c_{21}(M_2) = 2(c_{16}(M_2) + c_{17}(M_2) + c_{18}(M_2) + c_{19}(M_2) + M_2^2 + cc_9(M_2))$. Integrating (2.28) over $[0, \omega]$, we have

$$\int_0^\omega \|u_{Nxxxxxx}(\cdot, t)\|^2 dt \le 2c_{21}(M_2)\omega. \tag{2.29}$$

It then follows from (2.23) that there exists a time $t_5 \in (0, \omega)$ such that

$$\|u_{Nxxxxxx}(\cdot, t_5)\|^2 \le 2c_{21}(M_2). \tag{2.30}$$

Then

$$\begin{aligned} \|u_{Nxxxx}(\cdot, t_5)\|^2 &\le \frac{1}{16c}\|u_{Nxxxxxx}(\cdot, t_5)\|^2 + \frac{1}{c}c_{16}(M_2) \\ &\le \frac{1}{8c}c_{21}(M_2) + \frac{c_{16}(M_2)}{c}. \end{aligned} \tag{2.31}$$

Integrating (2.28) again over $[t_5, t+\omega]$ ($\forall t \in [0, \omega]$), using (2.31), we obtain

$$\sup_{0\le t\le\omega} \|u_{Nxxxx}(\cdot, t)\|^2 \le \|u_{Nxxxx}(\cdot, t_5)\|^2 + 2c_{21}(M_2)\omega = c_{22}(M_2). \tag{2.32}$$

Based on Sobolev's embedding theorem, we have

$$\|u_{Nxxx}(\cdot, t)\|_{C[0,1]} \le c\|u_N\|_{H^4} \le c_{22}(M_2), \quad t \in [0, \omega]. \tag{2.33}$$

Multiplying both sides of (2.1) by u_{Nt}, and integrating it over $(0, 1)$, we obtain

$$\begin{aligned} & \|u_{Nt}\|^2 \\ = \; & (-u_{Nxxxx} - (\frac{u_{Nx}}{1+|u_{Nx}|^2})_x + f, u_{Nt}) \\ \le \; & \frac{1}{2}\|u_{Nt}\|^2 + \frac{3}{2}\|u_{Nxxxx}\|^2 + \frac{3}{2}\|f\|^2 + \frac{3}{2}\int_0^1 \left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)_x^2 dx \\ \le \; & \frac{1}{2}\|u_{Nt}\|^2 + \frac{3}{2}\|u_{Nxxxx}\|^2 + \frac{3}{2}\|f\|^2 \\ & +3\int_0^1 \left(\frac{u_{Nxx}}{1+|u_{Nx}|^2}\right)^2 dx + 3\int_0^1 \left(\frac{2u_{Nx}^2 u_{Nxx}}{(1+|u_{Nx}|^2)^2}\right)^2 dx \\ \le \; & \frac{1}{2}\|u_{Nt}\|^2 + \frac{3}{2}\|u_{Nxxxx}\|^2 + \frac{3}{2}\|f\|^2 + c\|u_{Nxx}\|^2 \\ \le \; & \frac{1}{2}\|u_{Nt}\|^2 + \frac{3}{2}c_{22}(M_2) + \frac{3}{2}M_2^2 + cc_9(M_2). \end{aligned}$$

Hence

$$\|u_{Nt}\|^2 \le c_{23}(M_2) \equiv 3c_{22}(M_2) + 3M_2^2 + 2cc_9(M_2). \tag{2.34}$$

Differentiating (2.1) with respect to t, we get

$$u_{Ntt} + u_{Nxxxxt} + (\frac{u_{Nx}}{1+|u_{Nx}|^2})_{xt} = f_t, \quad (x,t) \in (0,1) \times \mathbf{R}. \tag{2.35}$$

Multiplying both sides of (2.35) by u_{Nxxt}, and integrating it over $(0,1)$, we obtain

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|u_{Nxt}\|^2 + \|u_{Nxxxt}\|^2 \\
= & \int_0^1 \left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)_t u_{Nxxxt}dx - \int_0^1 f_t u_{Nxxt}dx. \\
\le & \frac{1}{4}\|u_{Nxxxt}\|^2 + 2\int_0^1 \left(\frac{u_{Nt}}{1+|u_{Nx}|^2}\right)^2 dx \\
& +2\int_0^1 \left(\frac{2u_{Nx}^2 u_{Nxt}}{(1+|u_{Nx}|^2)^2}\right)^2 dx + \frac{1}{4}\|u_{Nxxt}\|^2 + M_2^2 \\
\le & \frac{1}{4}\|u_{Nxxxt}\|^2 + \frac{5}{2}\|u_{Nxt}\|^2 + \frac{1}{4}\|u_{Nxxt}\|^2 + M_2^2.
\end{aligned}$$

Using Nirenberg's inequality, we obtain

$$\frac{5}{2}\|u_{Nxt}\|^2 \le \frac{5}{2}(c_1'\|u_{Nxxxt}\|^{\frac{1}{3}}\|u_{Nt}\|^{\frac{2}{3}} + c_2'\|u_{Nt}\|)^2 \le \frac{1}{8}\|u_{Nxxxt}\|^2 + c_{24}(M_2),$$

and

$$\frac{1}{4}\|u_{Nxxt}\|^2 \le \frac{1}{4}(c_1'\|u_{Nxxxt}\|^{\frac{2}{3}}\|u_{Nt}\|^{\frac{1}{3}} + c_2'\|u_{Nt}\|)^2 \le \frac{1}{8}\|u_{Nxxxt}\|^2 + c_{25}(M_2).$$

Summing up, we have

$$\frac{d}{dt}\|u_{Nxt}\|^2 + \|u_{Nxxxt}\|^2 \le c_{26}(M_2), \tag{2.36}$$

where $c_{26}(M_2) = 2c_{24}(M_2) + 2c_{25}(M_2) + 2M_2^2$. Integrating (2.36) over $[0,\omega]$, we have

$$\int_0^\omega \|u_{Nxxxt}(\cdot,t)\|^2 dt \le c_{26}(M_2)\omega. \tag{2.37}$$

It then follows from (2.37) that there exists a time $t_6 \in (0,\omega)$ such that

$$\|u_{Nxxxt}(\cdot,t_6)\|^2 \le c_{26}(M_2). \tag{2.38}$$

Then

$$\|u_{Nxt}(\cdot,t_6)\|^2 \le \frac{1}{20}\|u_{Nxxxt}(\cdot,t_6)\|^2 + \frac{2}{5}c_{24}(M_2) \le \frac{1}{20}c_{26}(M_2) + \frac{2}{5}c_{24}(M_2). \tag{2.39}$$

Integrating (2.36) again over $[t_6, t+\omega]$ ($\forall t \in [0,\omega]$), using (2.39), we obtain

$$\sup_{0\le t\le\omega}\|u_{Nxt}(\cdot,t)\|^2 \le \|u_{Nxt}(\cdot,t_6)\|^2 + 2c_{26}(M_2)\omega = c_{27}(M_2). \tag{2.40}$$

Based on Sobolev's embedding theorem, we have

$$\|u_{Nt}(\cdot,t)\|_{C[0,1]} \leq c\|u_{Nt}\|_{H^1} \leq cc_{27}(M_2), \quad t\in[0,\omega]. \tag{2.41}$$

Multiplying both sides of (2.35) by u_{Nxxxxt}, and integrating it over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{Nxxt}\|^2 + \|u_{Nxxxxt}\|^2 = -\int_0^1 \left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)_{xt} u_{Nxxxxt}dx + \int_0^1 f_t u_{Nxxxxt}dx.$$

Therefore

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|u_{Nxxt}\|^2 + \|u_{Nxxxxt}\|^2 \\
\leq\ & \frac{1}{4}\|u_{Nxxxxt}\|^2 + 2\int_0^1 \left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)^2_{xt} dx + 2\int_0^1 f_t^2 dx \\
\leq\ & \frac{1}{4}\|u_{Nxxxxt}\|^2 + 8\int_0^1 \left(\frac{u_{Nxxt}}{1+|u_{Nx}|^2}\right)^2 dx + 8\int_0^1 \left(\frac{6u_{Nx}u_{Nxx}u_{Nxt}}{(1+|u_{Nx}|^2)^2}\right)^2 dx \\
& +8\int_0^1 \left(\frac{2u_{Nx}^2 u_{Nxxt}}{(1+|u_{Nx}^2)^2}\right) dx + 8\int_0^1 \left(\frac{4u_{Nx}^3 u_{Nxx}u_{Nxt}}{(1+|u_{Nx}|^2)^3}\right)^2 dx + 2\int_0^1 f_t^2 dx \\
\leq\ & \frac{1}{4}\|u_{Nxxxxt}\|^2 + c\|u_{Nxxt}\|^2 + c\|u_{Nxt}\|^2 + 2M_2^2.
\end{aligned}$$

Nirenberg's inequality gives

$$c\|u_{Nxxt}\|^2 \leq c(c_1'\|u_{Nxxxxt}\|^{\frac{1}{3}}\|u_{Nxt}\|^{\frac{2}{3}} + c_2'\|u_{Nxt}\|)^2 \leq \frac{1}{4}\|u_{Nxxxxt}\|^2 + c_{28}(M_2).$$

Summing up, we get

$$\frac{d}{dt}\|u_{Nxxt}\|^2 + \|u_{Nxxxxt}\|^2 \leq c_{29}(M_2), \tag{2.42}$$

where $c_{29}(M_2) = 2cc_{27}(M_2) + 2cc_{28}(M_2) + 4M_2^2$. Integrating (2.42) over $[0,\omega]$, we have

$$\int_0^\omega \|u_{Nxxxxt}(\cdot,t)\|^2 dt \leq c_{30}(M_2)\omega. \tag{2.43}$$

It then follows from (2.43) that there exists a time $t_7\in(0,\omega)$ such that

$$\|u_{Nxxxxt}(\cdot,t_7)\|^2 \leq c_{30}(M_2). \tag{2.44}$$

Then

$$\|u_{Nxxt}(\cdot,t_7)\|^2 \leq \frac{1}{4c}\|u_{Nxxxxt}(\cdot,t_7)\|^2 + \frac{c_{28}(M_2)}{c} \leq \frac{1}{4c}c_{30}(M_2) + \frac{c_{28}(M_2)}{c} \tag{2.45}$$

Integrating (2.42) again over $[t_7, t+\omega]$ $(\forall t\in[0,\omega])$, using (2.45), we obtain

$$\sup_{0\leq t\leq\omega} \|u_{Nxxt}(\cdot,t)\|^2 \leq \|u_{Nxxxxt}(\cdot,t_7)\|^2 + 2c_{30}(M_2)\omega = c_{31}(M_2). \tag{2.46}$$

Based on Sobolev's embedding theorem, we have

$$\|u_{Nxt}(\cdot,t)\|_{C[0,1]} \le c\|u_{Nt}\|_{H^2} \le cc_{21}(M_2), \quad t\in[0,\omega]. \tag{2.47}$$

Combining (2.9), (2.16), (2.21), (2.27), (2.33), (2.35), (2.40) and (2.46) together, we complete the proof of Lemma 2.2.

3 Existence and uniqueness of solutions for the problem (1.1)-(1.3)

Theorem 3.1 *Suppose that the assumptions in Lemma 2.2 is satisfied, then there exists a generalized time-periodic solution*

$$u(x,t)\in L^2_\omega(\mathbf{R};H^4(0,1)), \quad u_t(x,t)\in L^2_\omega(\mathbf{R};H^2(0,1)), \tag{3.1}$$

for problem (1.1)-(1.3), which satisfies

$$\int_0^\omega\int_0^1 \left(u_t+u_{xxxx}+(\frac{u_x}{1+|u_x|^2})_x-f\right)\varphi dxdt=0, \quad \forall\varphi\in L^2_\omega(\mathbf{R};L^2(0,1)). \tag{3.2}$$

Especially, if M_2 is sufficiently small, the solution is unique.

Proof. Based on Lemma 2.2 and Sobolev's embedding theorem, we obtain the following estimate

$$\sup_{0\le t\le\omega}(\|u_{Nt}(\cdot,t)\|_{C^1[0,1]}+\|u_N\|_{C^3[0,1]})\le c_{32}(M_2). \tag{3.3}$$

It then follows from (3.3) and Ascoli-Arzelá's theorem that there exists a function $u(x,t)$ and a subsequence of $\{u_N(x,t)\}$, still denoted by $\{u_N(x,t)\}$, such that, when $N\to+\infty$, $\{u_N(x,t)\}$, $u_{Nx}(x,t)\}$ uniformly converge to $u(x,t)$ and $u_x(x,t)$ on $[0,\omega]\times(0,1)$. Based on the result of Lemma 2.2, when $N\to+\infty$, the subsequences $\{u_{Nxx}\}$, $\{u_{Nxxx}\}$, $\{u_{Nxxxx}\}$, $\{u_{Nt}\}$, $\{u_{Nxt}\}$ and $\{u_{Nxxt}\}$ weakly converge to u_{xx}, u_{xxx}, u_{xxxx}, u_t, u_{xt} and u_{xxt} in $L^2_\omega(\mathbf{R};L^2(0,1))$. Set

$$W=\{u|u\in L^2_\omega(\mathbf{R};H^4(0,1)),u_t\in L^2_\omega(\mathbf{R};H^2(0,1))\}.$$

Aubin's compact lemma implies that the embedding $W\hookrightarrow L^2_\omega(\mathbf{R};H^2(0,1))$ is compact. Owing to the assumptions, we know that there exists a subsequence of $\{u_N(x,t)\}$ still denoted by $\{u_N(x,t)\}$ such that, when $N\to+\infty$, $\{u_N(x,t)\}$ is convergent in $L^2_\omega(\mathbf{R};H^3(0,1))$.

Setting $F(s)=\frac{s}{1+|s|^2}$, according to the previous subsequences $\{u_N(x,t)\}$, we conclude that $\{[F(u_{Nx})]_x\}=\{\left(\frac{u_{Nx}}{1+|u_{Nx}|^2}\right)_x\}$ weakly converges to $[F(u_x)]_x=\left(\frac{u_x}{1+|u_x|^2}\right)_x$ in $L^2_\omega(\mathbf{R};L^2(0,1))$. In fact, for any $\omega\in L^2_\omega(\mathbf{R};L^2(0,1))$, by (3.3), we have

$$
\begin{aligned}
& |\int_0^\omega ([F(u_{Nx})]_x - [F(u_x)]_x, w)dt| \\
\leq\ & \int_0^\omega\int_0^1 (|F'(u_{Nx}) - F'(u_x)||u_{Nxx}| + |F'(u_x)||u_{Nxx} - u_{xx}|)\,|\omega| dxdt \\
\leq\ & \int_0^\omega\int_0^1 (|F''(\theta u_{Nx} + (1-\theta)u_x)||u_{Nx} - u_x||u_{Nxx}||\omega|)\,dxdt \\
& + \int_0^\omega\int_0^1 (|F'(u_x)||u_{Nxx} - u_{xx}||\omega|)\,dxdt \\
\leq\ & c_{32}(M_2)\int_0^\omega\int_0^1 (|u_{Nx} - u_x| + |u_{Nxx} - u_{xx}|)|\omega| dxdt \\
\leq\ & c_{32}(M_2)[\|u_{Nx} - u_x\|_{L^2((0,\omega)\times(0,1))} + \|u_{Nxx} - u_{xx}\|_{L^2((0,\omega)\times(0,1))}] \\
& \cdot\|\omega\|_{L^2((0,\omega)\times(0,1))},
\end{aligned} \tag{3.4}
$$

where $\theta \in (0,1)$. By (3.4), we know that there exists a subsequence $\{u_N(x,t)\}$ such that $\{[F(u_{Nx})]_x\}$ weakly converges to $[F(u_x)]_x$ in $L^2(\mathbf{R}; L^2(0,1))$. Then, problem (1.1)-(1.3) admits a generalized time-periodic solution $u(x,t)$, which satisfies (3.1) and (3.2).

Now, we are going to prove the uniqueness of the solution. Suppose that $u(x,t)$ and $v(x,t)$ are two solutions of (1.1)-(1.3). Let $\xi(x,t) = u(x,t) - v(x,t)$, then $\xi(x,t)$ satisfies the following problem

$$
\begin{cases}
\xi_t + \xi_{xxxx} + [F(u_x)]_x - [F(v_x)]_x = 0, \quad x \in (0,1),\ t \in \mathbf{R}, \\
\xi_x(0,t) = \xi_x(1,t) = \xi_{xxx}(0,t) = \xi_{xxx}(1,t), \quad t \in \mathbf{R}, \\
\xi(t+\omega) = \xi(x,t), \quad t \in \mathbf{R}.
\end{cases} \tag{3.5}
$$

Multiplying both sides of the equation of (3.5) by ξ, integrating the products over $(0,1)$ and using the mean value theorem, we obtain that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \|\xi_{xx}\|^2 = \int_0^1 (F(u_x) - F(v_x))\xi_x dx \\
=\ & \int_0^1 (F'(\theta u_x + (1-\theta)v_x)\xi_{xx})\xi_x dx \leq c_{33}(M_2)\|\xi_{xx}\|\|\xi_x\| \\
\leq\ & \frac{1}{4}\|\xi_{xx}\|^2 + [c_{33}(M_2)]^2\|\xi_x\|^2 = \frac{1}{4}\|\xi_{xx}\|^2 - [c_{33}(M_2)]^2(\xi, \xi_{xx}) \\
\leq\ & \frac{1}{2}\|\xi_{xx}\|^2 + [c_{33}(M_2)]^4\|\xi\|^2.
\end{aligned} \tag{3.6}
$$

Using Poincaré's inequality, we get

$$
\|\xi\|^2 \leq \frac{1}{2}\|\xi_x\|^2 = -\frac{1}{2}(\xi, \xi_{xx}) \leq \frac{1}{4}\|\xi_{xx}\|^2 + \frac{1}{4}\|\xi\|^2,
$$

which means

$$
\|\xi\|^2 \leq \frac{1}{3}\|\xi_{xx}\|^2.
$$

It then follows form (3.6) and the above inequality that

$$\frac{d}{dt}\|\xi\|^2 + (3 - 2[c_{32}(M_2)]^4)\|\xi\|^2 \leq 0. \tag{3.7}$$

Taking M_2 sufficiently small such that $3 - 2[c_{32}(M_2)]^4 > 0$, using Gronwall's inequality, we have

$$\|\xi(\cdot, t)\|^2 \leq \|\xi(\cdot, 0)\|^2 e^{-(3-2[c_{32}(M_2)]^4)t}, \quad \forall t > 0.$$

Since $\xi(x, t)$ is time-periodic, for any $t \in \mathbf{R}$, there exists a natural number N_0 such that $t + N_0\xi > 0$ and

$$\|\xi(\cdot, t)\|^2 = \|\xi(\cdot, t + N_0\omega)\|^2 \leq \|\xi(\cdot, 0)\|^2 e^{-(3-2[c_{32}(M_2)]^4)N\omega}, \quad \forall N > N_0,$$

that is

$$\|\xi(\cdot, t)\|^2 = 0, \quad \forall t \in \mathbf{R}.$$

Then, Theorem 3.1 is proved.

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